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Convergence, Asymptotic Periodicity, and Finite-Point Blow-Up in One-Dimensional Semilinear Heat Equations

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We consider the initial value problem for the semilinear heat equation $u_t = u_{xx} + f(u, t)$ $(0 < x < L,\ t > 0)$ under the Dirichlet, the Neumann, or the periodic boundary conditions. We show that each solution—whether it exists globally for $t > 0$ or blows up in a finite time—possesses an "asymptotic profile" in a certain sense and tends to this profile as time increases. In the special case where $f(u, t+T) \equiv f(u, t)$ for some $T > 0$, among other things, the above statement is interpreted as saying that any bounded global solution converges as $t \to \infty$ to a time T-periodic solution having some specific spatial structure. In the case where the solution blows up in a finite time (say at $t = t_0$), assuming simply that f is a smooth function satisfying some growth conditions and that the initial data is a nonconstant bounded function, we prove that the blow-up set is a finite set and that $\lim_{t \uparrow t_0} u(x, t) = \varphi(x)$ exists, with φ being a smooth function having at most finitely many singular points. © 1989 Academic Press, Inc.

1. INTRODUCTION

In this paper we shall study the initial-boundary value problem for a semilinear parabolic equation of the form

$$u_t = u_{xx} + f(u, t), \qquad 0 < x < L,\ t > 0, \tag{1.1}$$

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under the initial condition

$$u(x, 0) = u_0(x), \qquad 0 \leqslant x \leqslant L \tag{1.2}$$

together with one of the following three types of boundary conditions:

(a) the Dirichlet boundary conditions

$$u(0, t) = u(L, t) = 0, \qquad t > 0; \tag{1.3a}$$

(b) the Neumann boundary conditions

$$u_x(0, t) = u_x(L, t) = 0, \qquad t > 0; \tag{1.3b}$$

(c) the periodic boundary conditions

$$\begin{aligned} u(0, t) &= u(L, t), \qquad t > 0, \\ u_x(0, t) &= u_x(L, t), \qquad t > 0. \end{aligned} \tag{1.3c}$$

Here f: $\mathbb{R} \times [0, \infty) \to \mathbb{R}$ is assumed to be smooth, and u_0 is continuous in $[0, L]$.

Our main interest is to investigate the asymptotic behavior of solutions to the above problems (1.1)–(1.3a), (1.1)–(1.3b), and (1.1)–(1.3c). We consider two cases: the case where the solution $u(x, t)$ exists globally in time (that is, for $0 \leqslant t < \infty$) and the case where it blows up in a finite time. In both cases it will be shown that each solution tends to some "asymptotic profile" as time increases, the meaning of which will be specified later.

To be more precise, denote by $u(x, t; \psi)$ the solution of problem (1.1)–(1.3) with initial data $u_0 = \psi$. Here and in what follows we shall not specify which of the boundary conditions (1.3a), (1.3b), and (1.3c) is considered, unless distinction is necessary. By the smoothness of f and the boundedness of u_0, the solution exists at least locally in time and is classical for $t > 0$. Here the initial condition (1.2) is understood in the sense that

$$u(x, t) \text{ remains bounded on } [0, L] \text{ as } t \downarrow 0 \text{ and } \lim_{t \downarrow 0} u(x, t) = u_0(x) \text{ locally uniformly in } x \in (0, L). \tag{1.2$'$}$$

It is well known that the convergence in (1.2)′ occurs in fact uniformly in $x \in [0, L]$ in the case of the boundary conditions (1.3b) and (1.3c). The same uniform convergence occurs in the case (1.3a) if $u_0(0) = u_0(L) = 0$. For each $\psi \in C([0, L])$, let

$$s(\psi) = \sup \left\{ s > 0 \;\middle|\; \begin{array}{l} \text{solution } u(x, t; \psi) \text{ can be} \\ \text{continued up to } t = s \end{array} \right\}. \tag{1.4}$$

Clearly $0 < s(\psi) \leqslant \infty$ and the existence theorem for local solutions of (1.1)–(1.3) implies

$$\lim_{t \uparrow s(\psi)} \|u(\cdot, t; \psi)\|_{L^\infty(0,L)} = \infty \tag{1.5}$$

if $s(\psi) < \infty$. In the next section we prove our fundamental lemma (Lemma A), which states that

$$\lim_{t \uparrow s(\psi)} \operatorname{sgn}(u_x(x, t; \psi))$$

exists for every $x \in [0, L]$, where $\operatorname{sgn}(\xi) = 1$, -1, or 0 depending on whether $\xi > 0$, < 0, or $= 0$. Roughly speaking, this lemma implies that the location of the local minima and the local maxima of the function $x \mapsto u(x, t)$ converges as $t \to s(\psi)$. A remarkable aspect of this lemma is that it holds true regardless of the nonlinearity $f(u, t)$ and no matter whether $s(\psi) < \infty$ or $s(\psi) = \infty$. A prototype of Lemma A was first introduced by Chen [9] to study semilinear heat equations on S^1, and its modified version has been used by Chen, Matano, and Véron [10, 11] in classifying isolated singularities of a semilinear elliptic equation in $\mathbb{R}^2$. This lemma can be proved by using the equivariance of Eq. (1.1) with respect to "reflection" and applying an argument similar to that in [22]. More precisely, for each parameter $a \in (0, L)$ we investigate how the number of zeroes of the function $x \mapsto u(2a - x, t) - u(x, t)$ changes as t increases, the study of which reaveals that $u_x(a, t)$ does not change sign if t is sufficiently close to $s(\psi)$. Lemma A will play an important role throughout the present paper.

Next we consider the case $s(\psi) = \infty$ and the case $s(\psi) < \infty$ separately and study these cases in more detail. First we deal with the case $s(\psi) = \infty$ and prove that, under certain regularity conditions on the nonlinear term f, any bounded time-global solution approaches as $t \to \infty$ a family of functions on $[0, L]$ sharing a common spatial symmetry property (Theorem B). In the special case where f is time-periodic, i.e., $f(u, t + T) \equiv f(u, t)$, it can further be shown that any bounded solution converges to a time-periodic solution as $t \to \infty$ (Theorem C). A result somewhat similar to Theorem C has recently been obtained by Alikakos and Hess [1] in a more abstract setting with applications to periodic-parabolic problems in several space dimensions, but their result is limited to those systems in which every orbit is Liapunov stable, which is a very strong requirement. Since our Theorem C assumes only that f be smooth and time-periodic, it is a far stronger result than that of [1] as far as one-dimensional problems of the form (1.1)–(1.3) are concerned.

An immediate corollary to Theorem C is that if $f \equiv f(u)$ is time-independent, or, in other words, if Eq. (1.1) is autonomous, then any bounded solution converges to an equilibrium solution as $t \to \infty$ (Theorem D). A more generalized version of Theorem D has been known in the case of the Dirichlet or the Neumann boundary conditions (1.3a), (1.3b) (see Matano [22], Zelenyak [31]), but our proof of Theorem D is quite different from that of [22] or [31], as we do not make use of a Liapunov functional associated with problem (1.1)–(1.3). Moreover the arguments in [22, 31] do not apply to the periodic boundary conditions (1.3c), while our method applies to any of the boundary conditions (1.3a), (1.3b) and (1.3c), although the spatial homogeneity of Eq. (1.1) is essential in our argument. In the case of the periodic boundary conditions (1.3c), a partial result has been obtained by Chen [9] and a fully generalized version by Matano [24]. The article [24] deals with equations of a more general form $u_t = u_{xx} + f(u, u_x)$. However, the argument in [24]—as well as those in [9, 22, 31]—is limited to autonomous equations, while our Theorem D is derived directly from Theorem C, which deals with nonautonomous equations.

Next we consider the case where $s(\psi) < \infty$, that is, the case where the solution blows up in a finite time. We are interested in the shape of the blow-up set, or, so to speak, the spatial location of the "hot-spots" at the explosion time. Assuming that the nonlinear term $f(u, t)$ is a rapidly (or at least "not too slowly") growing function in u in the sense to be specified later, we shall prove that there are only finitely many blow-up points if the initial state u_0 is taken to be spatially inhomogeneous (Theorems E, E′, and *F*). Moreover the number of the blow-up points does not exceed that of the local extremum points of $u_0(x)$ (or that of the local maximum points of $u_0(x)$ when positive solutions are concerned). Typical examples of the nonlinearities to which our theorems apply include $f(u) = u|u|^{q-1}$ $(q > 1)$, Ae^u $(A > 0)$, and $u(\log(1+u))^r$ $(r > 2)$. As regards the finiteness of the number of blow-up points, it is Weissler [30] who has first constructed an example of a single-point blow-up solution for problem (1.1)–(1.3a). Friedman and McLeod [15] (and also Mueller and Weissler [26]) have considered a wider class of nonlinearities f and proved that a single-point blow-up occurs for problem (1.1)–(1.3a) if $u_0(x)$ has a unique local maximum point and $f(u)$ satisfies certain growth conditions. They [15] have also obtained a similar result for the Neumann problem (1.1)–(1.3b) and for radially symmetric solutions of higher-dimensional problems (see also [16] for the one-dimensional Neumann problem). Caffarelli and Friedman [7] have improved the results in [15] for the Dirichlet problem (1.1)–(1.3a) by allowing $u_0(x)$ to have two local maxima and proved that the number of the blow-up points is at most two. (Y.-G. Chen [12] deals with the higher-dimensional version of [7] for radially symmetric solutions.) As far as one-dimensional problems are concerned, our theorems in the present paper (Theorem E, E′, and F) are far stronger than those in [7, 15, 16, 26, 30], since we do not impose any restriction on the number of local maxima or minima of the initial data u_0, and in fact it can be infinite initially. (A preliminary version of Theorem F can be found in [24].)

This paper is organized as follows: In Section 2 we prove Lemma A by using what can be called a "reflection method." In Section 3 we consider bounded global solutions and prove Theorem B. We deal with periodic problems in Section 4 (Theorem C) and autonomous problems in Section 5 (Theorem D). Finally, we consider the blow-up problem in Section 6 and prove Theorems E, E′, and F.

2. Fundamental Lemma

In this section we assume that f: $\mathbb{R} \times [0, \infty) \to \mathbb{R}$ satisfies the following:

(A.0) *$f(u, t)$ is locally Hölder continuous in $\mathbb{R} \times [0, \infty)$ and continuously differentiable with respect to u.*

This assumption ensures the existence of a local solution to problem (1.1)–(1.3). No other assumption on f is made in this section, as far as the boundary conditions (1.3b) and (1.3c) are concerned. In the case of the boundary conditions (1.3a), we assume that:

(A.1) *One of the following conditions holds*:

(i) $f(-u, t) \equiv -f(u, t)$;

(ii) $u_0(x) \geqslant 0$ *and* $f(0, t) \equiv 0$;

(iii) $u_0(x) \geqslant 0$ *and* $f(0, t) \geqslant 0$; *moreover* $u_0(x)$ *is real analytic in* $[0, L]$ *or* $f(u, t)$ *is real analytic in* $[0, \infty) \times [0, \infty)$.

Define

$$\operatorname{sgn}(\xi) = \begin{cases} 1, & (\xi > 0), \\ 0, & (\xi = 0), \\ -1, & (\xi < 0). \end{cases} \tag{2.1}$$

The main result in this section is the following:

Lemma A. *Let* (A.0) *hold, and assume that* $u_0(x)$ *is continuous on* $[0, L]$. *In the case of the Dirichlet boundary conditions* (1.3a), *we further assume* (A.1). *Let* $u(x, t)$ *be the solution of* (1.1)–(1.3). *Then*

$$\lim_{t \uparrow s(u_0)} \operatorname{sgn}(u_x(x, t)) \tag{2.2}$$

exists for every $x \in [0, L]$, *where* $s(u_0)$ *is as defined in* (1.4).

Lemma A will be proved by converting problem (1.1)–(1.3) into a problem on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ of the form

$$u_t = u_{xx} + f(u, t), \qquad x \in S^1, t > 0, \tag{2.3}$$

$$u(x, 0) = u_0(x), \qquad x \in S^1. \tag{2.4}$$

We need some notations and preliminary lemmas: First, we denote by $C(S^1)$ the space of all continuous functions w: $S^1 \to \mathbb{R}$ endowed with the norm

$$\|w\|_{C(S^1)} = \|w\|_{L^\infty(S^1)} = \max_{x \in S^1} |w(x)|$$

and by $C^1(S^1)$ the space of all continuously differentiable functions w: $S^1 \to \mathbb{R}$ endowed with the norm

$$\|w\|_{C^1(S^1)} = \max_{x \in S^1} (|w(x)| + |w'(x)|).$$

Notation 2.1. Given a continuous function $w(x)$ on S^1, we define the *nodal number* of w by

$$\nu(w) = \text{the number of points } x \in S^1 \text{ with } w(x) = 0.$$

This defines the functional ν: $C(S^1) \to \mathbb{N} \cup \{0\} \cup \{\infty\}$.

DEFINITION 2.2. We say that $w \in C^1(S^1)$ possesses only simple zeroes, if $w'(x) \neq 0$ for any $x \in S^1$ such that $w(x) = 0$. The set of all such functions is denoted by Σ.

LEMMA 2.3. *Σ is open and everywhere dense in the space $C^1(S^1)$. The nodal numbers $\nu(w)$ is finite if $w \in \Sigma$, and the functional of nodal numbers $\nu|_\Sigma$: $\Sigma \to \mathbb{N} \cup \{0\}$ is locally constant.*

The proof is easy and is omitted.

LEMMA 2.4. *Let $q(x, t)$ and $r(x, t)$ be locally bounded functions on $S^1 \times (t_0, t_1)$ with q_x, q_t both locally bounded, and let $w(x, t)$ be a classical solution of*

$$w_t = w_{xx} + q(x, t)\, w_x + r(x, t)\, w, \qquad x \in S^1, t \in (t_0, t_1). \tag{2.5}$$

Assume that w is not identically equal to zero. Then

(i) *$\nu(w(\cdot, t))$ is finite for any $t \in (t_0, t_1)$ and is monotone nonincreasing in t;*

(ii) *For each* $t^* \in (t_0, t_1)$, $w(\cdot, t)$ *belongs to* Σ *for any* $t \in [t^*, t_1)$ *except for at most finitely many points* $s_1, s_2, ..., s_k \in [t^*, t_1)$;

(iii) *If* $w(\cdot, t^*) \notin \Sigma$ *for some* $t^* \in (t_0, t_1)$, *then*

$$v(w(\cdot, t)) > v(w(\cdot, s))$$

for any $t \in (t_0, t^*)$ *and* $s \in (t^*, t_1)$.

The above lemma is due to Angenent [3]. Note that the statement (ii) is a direct consequence of (i) and (iii). The nonincreasing property of $v(w(\cdot, t))$ in statement (i) is well known; the important point is the finiteness of $v(w(\cdot, t))$. A prototype of statement (iii) is given by Angenent and Fiedler [4] in the special case where the solution $w(x, t)$ is analytic in x, t. In an earlier work, Matano has proved a result similar to (but weaker than) statement (ii) [23; the proof of Lemma 5.4].

Notation 2.5. For convenience, we introduce the following notation: given a function w on S^1 and a point $a \in S^1$, we define functions $\sigma_a w$ and $\rho_a w$ on S^1 as below:

$$(\sigma_a w)(x) = w(x + a),$$

$$(\rho_a w)(x) = w(2a - x).$$

The operators $w \mapsto \sigma_a w$ and $w \mapsto \rho_a w$ are called *shift* and *reflection*, respectively.

Note that Eq. (2.3) is equivariant with respect to these operators. In other words, if $u(x, t)$ is a solution of (2.3), then so are $\sigma_a u$ and $\rho_a u$. This fact will play an important role throughout the present paper. In particular, in the proof of Lemma A, the equivariance of (2.3) with respect to the reflection is essential.

Proof of Lemma A. First we prove the conclusion of the lemma for the case where $u(x, t)$ is a solution of problem (2.3)–(2.4). Let $w = \rho_a u - u$. Since both u and $\rho_a u$ are solutions to Eq. (2.3), $w(x, t)$ satisfies a linear parabolic equation of the form (2.5) with $t_0 = 0$, $t_1 = s(u_0)$, $q(x, t) \equiv 0$, and

$$r(x, t) = \{f(\rho_a u, t) - f(u, t)\}/(\rho_a u - u); \tag{2.6}$$

r is a locally bounded function by the assumption on f. Moreover

$$\left.\begin{aligned} w(a, t) &= 0, \\ w_x(a, t) &= -2u_x(a, t), \end{aligned}\right. \qquad 0 \leqslant t < s(u_0). \tag{2.7}$$

If $w \equiv 0$, then $u_x(a, t) \equiv 0$ for $t \in [0, s(u_0))$ and the conclusion of the lemma is trivial. So we consider the case where $w \not\equiv 0$. Obviously $w_x(a, t) = 0$

implies $w(\cdot, t) \notin \Sigma$. It follows from Lemma 2.4(ii) that for any $0 < t^* < s(u_0)$, $w_x(a, t)$ vanishes only at (at most) finitely many points in the interval $[t^*, s(u_0))$. Consequently, $w_x(a, t) \neq 0$ for $t^* \leqslant t < s(u_0)$ if t^* is chosen sufficiently close to $s(u_0)$. In particular, $w_x(a, t) = -2u_x(a, t)$ does not change sign in $t^* \leqslant t < s(u_0)$. This shows that the limit (2.2) exists if u is a solution of problem (2.3)–(2.4).

Now we come back to the case where u is a solution of problem (1.1)–(1.3). In the case of the periodic boundary conditions (1.3c), problem (1.1)–(1.3) can naturally be converted into the form (2.3)–(2.4). In the case of the Neumann boundary conditions (1.3b), by extending the function $u(x, t)$ as

$$\tilde{u}(x, t) = \begin{cases} u(x, t), & 0 \leqslant x \leqslant L, \\ u(-x, t), & -L \leqslant x < 0, \end{cases} \tag{2.8}$$

we obtain a periodic boundary value problem on $[-L, L]$, hence the problem can again be converted into the form (2.3)–(2.4). Finally, we consider the case (1.3a). If assumption (A.1)(i) holds, then letting

$$\tilde{u}(x, t) = \begin{cases} u(x, t), & 0 \leqslant x \leqslant L, \\ -u(-x, t), & -L \leqslant x < 0, \end{cases} \tag{2.9}$$

we obtain a periodic boundary value problem on $[-L, L]$, hence the problem is equivalent to (2.3)–(2.4). In the case where assumption (A.1)(ii) holds, replacing $f(u, t)$ by

$$\tilde{f}(u, t) = \begin{cases} f(u, t), & u \geqslant 0, \\ -f(-u, t), & u < 0, \end{cases}$$

we can reduce the problem to the case where (A.1)(i) holds.

It remains to consider the case (1.3a) under assumption (A.1)(iii). In this case, the problem, in general, can no longer be converted into the form (2.3)–(2.4), so we have to use a different argument. We only give a sketch of the proof. There are two cases to be considered: the case where $u_0(x)$ is real analytic in x and the case where $f(u, t)$ is real analytic in u, t. In the latter case, $u(x, t)$ is real analytic in $x \in [0, L]$ for each $0 < t < s(u_0)$ (see [14, 19]), so this reduces to the former case. In what follows we assume that u_0 is real analytic on $[0, L]$. We only consider the case where $0 < a < L$, since the case $a = 0$ or $a = L$ can be treated much more easily. We define the function $w = \rho_a u - u$ in the region $a \leqslant x \leqslant L^*$, $0 \leqslant t < s(u_0)$, where $L^* = \min\{L, 2a\}$. First suppose that $w(x, 0) \not\equiv 0$ in $[a, L^*]$. Since $u_0(x)$ is assumed to be real analytic, $w(x, 0)$ is real analytic on $[a, L^*]$, so it has only finitely many zeroes. Moreover w satisfies a linear parabolic equation of the form (2.5) in the region $(a, L^*)\times(0, s(u_0))$, together with either of the boundary conditions

$$w(a, t)=0, \qquad w(L^*, t)\geqslant 0, \qquad t>0 \tag{2.10a}$$

or

$$w(a, t)=0, \qquad w(L^*, t)\leqslant 0, \qquad t<0, \tag{2.10b}$$

depending on whether $0<a<L/2$ or $L/2\leqslant a<L$. Combining these observations and using an argument similar to that in [22], one sees that there exist $t_0\in[0, s(u_0))$ and a continuous function $\xi(t)$: $[t_0, s(u_0))\to(a, L^*]$ such that $w(x, t)$ does not change sign in the region $\{(x, t)|\ t_0\leqslant t<s(u_0), a<x<\xi(t)\}$. From the strong maximum principle, (2.10), and the fact that $w\not\equiv 0$, it follows that either $w_x(a, t)>0$ for $t_0<t<s(u_0)$ or $w_x(a, t)<0$ for $t_0<t<s(u_0)$. Hence the convergence in (2.2) follows. Since the case $w(x, 0)\equiv 0$ can be treated more easily, we omit the proof for this case. This completes the proof of the lemma.

3. Behavior of Bounded Solutions

In this section we consider the case where the solution $u(x, t)$ of (1.1)–(1.3) stays bounded (in $L^\infty(0, L)$) as $t\uparrow s(u_0)$. In this case the solution exists globally in time—that is, $s(u_0)=\infty$—by virtue of (1.5). Throughout this section we assume the following:

(A.2)(i) *For each positive number $M>0$ and $T\geqslant 0$, $f(u, t)$ is Hölder continuous (of exponent $(\alpha, \alpha/2)$, $0<\alpha<1$) in the region $[-M, M]\times[T, T+1]$ and its Hölder norm*

$$\sup_{\substack{|u|,|w|\leqslant M\\ s,t\in[T,T+1]\\ (w,s)\neq(u,t)}}\frac{|f(w, s)-f(u, t)|}{|w-u|^\alpha+|s-t|^{\alpha/2}}$$

is bounded by a constant independent of T;

(ii) *for each $M>0$, $f(u, t)$ is uniformly Lipschitz continuous in u in the region $[-M, M]\times[0, \infty)$.*

Although Eq. (1.1) is not autonomous, for convenience sake we use the terminology "ω-limit set" in order to discuss the asymptotic behavior of solutions.

Definition 3.1. Given $\psi\in C([0, L])$, let $u(x, t; \psi)$ be the solution of (1.1)–(1.3) with initial data $u_0=\psi$ and suppose $s(\psi)=\infty$. We define the *ω-limit set* of ψ by

$$\omega(\psi)=\bigcap_{t>0}\operatorname{closure}\{u(\cdot, \tau)\,|\,\tau\geqslant t\},$$

where the closure is with respect to the topology of $C[0, L]$).

By assumption (A.2) and standard a priori estimates, one easily sees that $\omega(\psi)$ remains unchanged if the "closure" is taken in the topology of $C^2([0, L])$ instead of $C([0, L])$, and that it is a nonempty compact connected set in $C^2([0, L])$ if $u(\cdot, t; \psi)$ remains bounded in $C([0, L])$ as $t \to \infty$.

DEFINITION 3.2. Let $w \in C^1(S^1)$. We call w a *symmetrically oscillating function* if there exist $x_0 \in S^1$ and $m \in \mathbb{N}$ such that

$$w(x) = w(2x_0 - x), \qquad x \in S^1; \tag{3.1a}$$

$$w'(x) > 0, \qquad x \in (x_0, x_0 + 1/2m); \tag{3.1b}$$

$$w(x + 1/m) = w(x), \qquad x \in S^1. \tag{3.1c}$$

We denote the set of all such functions by $\mathfrak{G}_m(x_0)$.

Qualitatively, the symmetrically oscillating functions in $\mathfrak{G}_m(x_0)$ look like the function $-\cos 2\pi m(x - x_0)$.

DEFINITION 3.3. Let $w \in C^1([0, L])$ be such that $w(0) = w(L)$ and $w'(0) = w'(L)$. We call w a *symmetrically oscillating function under the periodic boundary conditions* if there exist $x_0 \in [0, L]$ and $m \in \mathbb{N}$ such that $\tilde{w} \in \mathfrak{G}_m([x_0/L])$, where $x \mapsto [x]$ denotes the canonical homomorphism $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ and $\tilde{w}$ denotes the function on S^1 determined uniquely by the relation

$$\tilde{w}([x/L]) = w(x), \qquad 0 \leqslant x \leqslant L.$$

We denote by $\mathfrak{G}^P_m(x_0)$ the set of all such functions.

DEFINITION 3.4. Let $w \in C^1([0, L])$ be such that $w'(0) = w'(L) = 0$. We call w a *symmetrically oscillating function under the Neumann boundary conditions* if the function $\tilde{w}(x) = w(L - |2x - L|)$ belongs to either $\mathfrak{G}^P_m(0)$ or $\mathfrak{G}^P_m(L/2m)$ for some $m \in \mathbb{N}$. In the former case, the set of all functions satisfying the above condition will be denoted by $\mathfrak{G}^N_{m,+}$ and in the latter case by $\mathfrak{G}^N_{m,-}$.

DEFINITION 3.5. Let $w \in C^1([0, L])$ be such that $w(0) = w(L) = 0$. We call w a *symmetrically oscillating function under the Dirichlet boundary conditions* if the function $\tilde{w}(x) = \mathrm{sgn}(L - 2x) \cdot w(L - |2x - L|)$ belongs to either $\mathfrak{G}^P_m(3L/4m)$ or $\mathfrak{G}^P_m(L/4m)$ for some $m \in \mathbb{N}$. In the former case, the set of all functions satisfying the above condition is denoted by $\mathfrak{G}^D_{m,+}$ and in the latter case by $\mathfrak{G}^D_{m,-}$.

Qualitatively, the symmetrically oscillating functions in $\mathfrak{G}^P_m(x_0)$ look like the function $-\cos 2m\pi(x - x_0)/L$, those in $\mathfrak{G}^N_{m,\pm}$ like $\mp\cos m\pi x/L$, and those in $\mathfrak{G}^D_{m,\pm}$ like $\pm\sin m\pi x/L$.

THEOREM B. *Let $f(u, t)$ be C^1 in u and let* (A.2) *hold. In the case of the Dirichlet boundary conditions* (1.3a), *assume further* (A.1)(i) *or* (ii). *Let u be a solution of* (1.1)–(1.3) *such that $\|u(\cdot, t)\|_{C([0, L])}$ remains bounded as $t \to \infty$. Then $\omega(u_0)$ is a nonempty compact connected subset of $C^2([0, L])$. Moreover, in the case of the periodic boundary conditions* (1.3c), *there exist $x_0 \in [0, L]$ and $m \in \mathbb{N}$ such that*

$$\omega(u_0) \subset \mathfrak{G}_m^P(x_0) \cup \mathfrak{C}, \tag{3.2}$$

where $\mathfrak{C}$ is the set of all constant functions on $[0, L]$. In the case of the Dirichlet boundary conditions (1.3a) *or the Neumann boundary conditions* (1.3b), *there exists $m \in \mathbb{N}$ such that either*

$$\omega(u_0) \subset \mathfrak{G}_{m,+}^* \cup \mathfrak{C} \tag{3.3a}$$

or

$$\omega(u_0) \subset \mathfrak{G}_{m,-}^* \cup \mathfrak{C}, \tag{3.3b}$$

*where * stands for either D or N depending on which of the boundary conditions* (1.3a) *and* (1.3b) *is considered.*

Remark 3.6. A well-known sufficient condition for the boundedness of solutions is the existence of a constant $M > 0$ such that

$$u \cdot f(u, t) \leqslant 0$$

for $|u| \geqslant M$, $t \geqslant 0$. In the case of the Dirichlet boundary conditions (1.3a), the above condition can be weakened as

$$\limsup_{|u| \to \infty} \left[\sup_{t \geqslant 0} \frac{f(u, t)}{u} \right] < \pi^2/L^2.$$

LEMMA 3.7. *Let u be a solution of* (2.3)–(2.4) *that remains bounded as $t \to \infty$ and let φ be an arbitrary element of $\omega(u_0)$. Then for each $a \in S^1$, we have*

(i) *either $\rho_a \varphi = \varphi$ or $\rho_a \varphi - \varphi \in \Sigma$;*

(ii) *$\rho_a \varphi = \varphi$ if and if $\varphi'(a) = 0$.*

Proof. Take a sequence $t_k \to \infty$ such that $u(\cdot, t_k) \to \varphi$ in $C^1(S^1)$. Define $f_k(u, t) = f(u, t + t_k)$ $(k = 1, 2, 3, \ldots)$. Then for every $k_0 \in \mathbb{N}$ and $M \geqslant 0$, the sequence of functions $\{f_k\}_{k \geqslant k_0}$ is relatively compact in $C_{\text{loc}}^{\beta, \beta/2}([-M, M] \times [-t_{k_0}, \infty))$, where β is any number in $(0, \alpha)$ with α as in assumption (A.2). (Here and in what follows $C_{\text{loc}}^{\beta, \beta/2}$ denotes the space of functions that are locally Hölder continuous of exponent $(\beta, \beta/2)$.) Replacing $\{t_k\}$ by its subsequence if necessary, we may assume without loss of generality that f_k

converges as $k \to \infty$ in the topology of $C_{\text{loc}}^{\beta, \beta/2}(\mathbb{R} \times \mathbb{R})$. Let $f_\infty = \lim_{k \to \infty} f_k$. By the second condition in (A.2), for each $M > 0$, $f_\infty(u, t)$ is uniformly Lipschitz continuous in u in the region $[-M, M] \times \mathbb{R}$. Similarly, we define $u_k(x, t) = u(x, t + t_k)$, $k = 1, 2, 3, \ldots$. By assumption (A.2) and the Schauder estimates, the sequences $\{u_k\}_{k \geqslant k_0}$, $\{\partial u_k/\partial t\}_{k \geqslant k_0}$, $\{\partial u_k/\partial x\}_{k \geqslant k_0}$, and $\{\partial^2 u_k/\partial x^2\}_{k \geqslant k_0}$ are relatively compact in $C_{\text{loc}}^{\beta, \beta/2}(S^1 \times (-t_{k_0}, \infty))$. We can therefore choose a subsequence, denoted again by $\{u_k\}$, converging to a function, say p, in the following sense:

$$\begin{aligned} &u_k(x, t) \to p(x, t), & &\partial u_k/\partial t(x, t) \to \partial p/\partial t(x, t), \\ &\partial u_k/\partial x(x, t) \to \partial p/\partial x(x, t), & &\partial^2 u_k/\partial x^2(x, t) \to \partial^2 p/\partial x^2(x, t) \end{aligned} \tag{3.4}$$

locally uniformly in $S^1 \times \mathbb{R}$. It is easily seen that the limit function p satisfies

$$p_t = p_{xx} + f_\infty(p, t), \qquad x \in S^1,\ t \in \mathbb{R}, \tag{3.5a}$$

$$p(x, 0) = \varphi(x), \qquad x \in S^1. \tag{3.5b}$$

Defining $w = \rho_a p - p$, we find that

$$\begin{aligned} w_t &= w_{xx} + r_\infty(x, t)\, w, & &x \in S^1,\ t \in \mathbb{R}, \\ w(x, 0) &= (\rho_a \varphi - \varphi)(x), & &x \in S^1, \end{aligned}$$

where r_∞ is a bounded function.

Assume that $\rho_a \varphi \neq \varphi$. From Lemma 2.4(ii), there exists $\delta > 0$ such that $w(\cdot, \delta)$ and $w(\cdot, -\delta)$ both lie in Σ. In view of this, and using Lemma 2.3 and (3.4), we see that there exists an integer N such that

$$v((\rho_a u - u)(\cdot, t_k + \delta)) = v(w(\cdot, \delta)) \tag{3.6}$$

for any $k \geqslant N$. But $\rho_a u - u$ satisfies a parabolic equation of the form (2.5) with $q \equiv 0$ and r locally bounded, so we conclude from Lemma 2.4(i) and (3.6) that

$$v((\rho_a u - u)(\cdot, t)) = v(w(\cdot, \delta)), \qquad t \geqslant t_N + \delta. \tag{3.7}$$

Similarly, choosing integer K sufficiently large, we have

$$v((\rho_a u - u)(\cdot, t)) = v(w(\cdot, -\delta)), \qquad t \geqslant t_K - \delta. \tag{3.8}$$

It follows that $v(w(\cdot, \delta)) = v(w(\cdot, -\delta))$. By Lemma 2.4(iii), this implies that $w(\cdot, 0) = \rho_a \varphi - \varphi \in \Sigma$. Thus the assertion (i) of the lemma is proved. Since the assertion (ii) follows immediately from (i), the proof of Lemma 3.7 is complete.

LEMMA 3.8. *Let u be as in Lemma* 3.7. *Then*

$$\omega(u_0) \subset \mathfrak{C} \cup \{\mathfrak{G}_m(x) \mid x \in S^1, m \in \mathbb{N}\}. \tag{3.9}$$

Proof. Let φ be an element of $\omega(u_0) \backslash \mathfrak{C}$. Choose $x_0 \in S^1$ to be a minimum point of φ:

$$\varphi(x_0) = \min_{x \in S^1} \varphi(x).$$

Since φ is nonconstant, the set $\{x \in S^1 \mid \varphi'(x) = 0\}$ is discrete by virtue of Lemma 3.7(ii). It is therefore possible to find $x_1 \in S^1$ such that

$$\begin{aligned} &\varphi'(x) > 0, \qquad x \in (x_0, x_1); \\ &\varphi'(x_1) = 0, \end{aligned}$$

where $(x_0, x_1) = \{x_0 + [\theta d] \mid 0 < \theta < 1\}$ with $d \in [0, 1)$ being defined by $[d] = x_1 - x_0$. By Lemma 3.7(ii) we have $\rho_{x_0}\varphi = \rho_{x_1}\varphi = \varphi$. It then easily follows that $d = 1/2m$ for some $m \in \mathbb{N}$ and that $\varphi \in \mathfrak{G}_m(x_0)$. This proves Lemma 3.8.

Proof of Theorem B. Arguing as in the proof of Lemma A, we can reduce problem (1.1)–(1.3) to the form (2.3)–(2.4). Let $\tilde{u}(x, t)$ be a solution to problem (2.3)–(2.4) corresponding to the solution $u(x, t)$ of (1.1)–(1.3). From Lemma A, we see that if $\varphi, \tilde{\varphi}$ are an arbitrary pair of elements of $\omega(\tilde{u}_0)$ and if $\varphi'(a) > 0$ for some $a \in S^1$, then $\tilde{\varphi}'(a) \geqslant 0$. Combining this with Lemma 3.8, we see that

$$\omega(\tilde{u}_0) \subset \mathfrak{G}_m(x_0) \cup \mathfrak{C}$$

for some $x_0 \in S^1$ and $m \in \mathbb{N}$. The original assertions (3.2) and (3.3) then follow immediately. The theorem is proved.

Remark 3.9. The conclusion of Theorem B remains true under condition (A.1)(iii) (plus the Dirichlet conditions (1.3a)), although in this case the problem in general cannot be convert into the form (2.3)–(2.4). In fact, by considering the function $\rho_a\varphi - \varphi$ on the interval $[a, L^*]$, where $L^* = \min\{2a, L\}$, and using the positivity of solution u, one can prove an analogue of Lemma 3.7, from which it follows that $\omega(u_0) \subset \mathfrak{G}^D_{1,+} \cup \{0\}$. In other words, for each $\varphi \in \omega(u_0) \backslash \{0\}$, it holds that

$$\begin{aligned} &\varphi(0) = \varphi(L) = \varphi'(L/2) = 0, \\ &\varphi'(x) > 0, \qquad 0 \leqslant x < L/2, \\ &\varphi(L - x) = \varphi(x), \qquad 0 \leqslant x \leqslant L. \end{aligned}$$

This remark applies also to Theorem C in the next section.

4. Periodic Problems

In this section we consider the case where Eq. (1.1) is periodic in t. Our aim is to show that any bounded global solution of a time-periodic problem converges to a periodic solution as $t \to \infty$. We assume the following:

(A.3) $f(u, t) \equiv f(u, t+T)$ *for some* $T > 0$.

The main result in this section is as follows:

Theorem C. *Let* (A.0) *and* (A.3) *hold. In the case of the Dirichlet boundary conditions* (1.3a), *assume further that* (A.1)(i) *or* (A.1)(ii) *holds. Let* $u(x, t)$ *be a solution of* (1.1)–(1.3) *such that* $\|u(\cdot, t)\|_{C([0, L])}$ *remains bounded as* $t \to \infty$. *Then there exists a solution* $p(x, t)$ *of* (1.1)–(1.3) *with* $p(x, t+T) \equiv p(x, t)$ *such that*

$$\lim_{t \to \infty} \|u(\cdot, t) - p(\cdot, t)\|_{C^2([0, L])} = 0. \tag{4.1}$$

Moreover, for each $t \in \mathbb{R}$, $p(\cdot, t)$ *belongs to a family of symmetrically oscillating functions on* $[0, L]$ *as specified in Theorem* B.

Definition 4.1. *Let* λ: $\mathbb{R} \to \mathbb{R}/T\mathbb{Z}$ denote the canonical homomorphism. Let $u(x, t; \psi)$ be the solution of (2.3)–(2.4) with initial data $u_0 = \psi \in C(S^1)$ and suppose $s(\psi) = \infty$. For each $\tau \in \mathbb{R}/T\mathbb{Z}$, we define

$$\omega(\psi; \tau) = \bigcap_{n \in \mathbb{N}} \text{closure}\{u(\cdot, t+kT; \psi) | k \geqslant n\}, \tag{4.2}$$

where the closure is with respect to the topology of $C^1(S^1)$ and t is a real number satisfying $\lambda(t) = \tau$.

It is clear that the right-hand side of (4.2) is independent of the choice of $t \in \lambda^{-1}(\tau)$ and therefore the set $\omega(\psi; \tau)$ is well defined. Obviously we have $\omega(\psi; \tau) \subset \omega(\psi)$ for every $\tau \in \mathbb{R}/T\mathbb{Z}$; moreover it is not difficult to check that

$$\omega(u_0) = \bigcup_{\tau \in \mathbb{R}/T\mathbb{Z}} \omega(u_0; \tau). \tag{4.3}$$

Lemma 4.2. *Let f satisfy* (A.0) *and* (A.3). *Let u be a bounded solution of* (2.3)–(2.4), *and let* $\tau_0 \in \mathbb{R}/T\mathbb{Z}$. *Finally, let* $t_0 \in \mathbb{R}$ *be such that* $\lambda(t_0) = \tau_0$. *Then* *for any* $\varphi \in \omega(u_0; \tau_0)$ *there exists a function* $p(x, t)$ *defined on* $S^1 \times \mathbb{R}$ *satisfying*

$$p_t = p_{xx} + f(p, t), \qquad x \in S^1, t \in \mathbb{R}, \tag{4.4a}$$

$$p(x, t_0) = \varphi(x), \qquad x \in S^1, \tag{4.4b}$$

$$p(\cdot, t) \in \omega(u_0; \lambda(t)), \qquad t \in \mathbb{R}. \tag{4.4c}$$

Proof. The lemma follows easily from (3.4) and (3.5) in the proof of Lemma 3.7 and (A.3).

LEMMA 4.3. *Let* $\varphi \in \omega(u_0; \tau_0)$ *and* p *be as in Lemma* 4.2. *Then*

$$p(x, t+T) \equiv p(x, t), \qquad x \in S^1, t \in \mathbb{R}. \tag{4.5}$$

Proof. Suppose the contrary:

$$p(x, t+T) \not\equiv p(x, t). \tag{4.6}$$

We shall derive a contradiction.

By the uniqueness theorem and the backward uniqueness theorem for the parabolic equation (4.4a) (see, for instance, Friedman [13]), (4.6) implies that $p(\cdot, t+T) \neq p(\cdot, t)$ for any $t \in \mathbb{R}$. It follows from this, Theorem B, and (4.4c) that $p(\cdot, t+T) \neq \sigma_a p(\cdot, t)$ for any $a \in S^1$ and $t \in \mathbb{R}$, where σ_a is the shift operator defined in Notation 2.5. By Lemma 2.4(ii), for any fixed $t \in \mathbb{R}$ and $a \in S^1$, there exists some $\delta > 0$ such that

$$p(\cdot, t+\delta+T) - \sigma_a p(\cdot, t+\delta) \in \Sigma,$$
$$p(\cdot, t-\delta+T) - \sigma_a p(\cdot, t-\delta) \in \Sigma.$$

Considering that $p(\cdot, t \pm \delta)) \in \omega(u_0; \lambda(t \pm \delta))$ and using Lemmas 2.3 and 2.4(i), we obtain

$$v(p(\cdot, t \pm \delta + T) - \sigma_a p(\cdot, t \pm \delta)) = \lim_{s \to \infty} v(u(\cdot, s+T) - \sigma_a u(\cdot, s)),$$

hence

$$v(p(\cdot, t+\delta+T) - \sigma_a p(\cdot, t+\delta)) = v(p(\cdot, t-\delta+T) - \sigma_a p(\cdot, t-\delta)).$$

It follows from this and Lemma 2.4(iii) that

$$p(\cdot, t+T) - \sigma_a p(\cdot, t) \in \Sigma, \qquad t \in \mathbb{R}, a \in S^1, \tag{4.7}$$

therefore, by Lemma 2.3,

$$v(p(\cdot, t+T) - \sigma_a p(\cdot, t)) = v_0, \qquad t \in \mathbb{R}, a \in S^1, \tag{4.8}$$

where ν_0 is some nonnegative integer independent of t and a. Since $p(\cdot, t_0) \in \omega(u_0; \tau_0)$ with t_0 being as in Lemma 4.2, we can choose a sequence $t_n \to \infty$ satisfying

$$t_n + T < t_{n+1} \qquad \text{and} \qquad t_n \equiv t_0 (\text{mod } T), \tag{4.9}$$

$$u(\cdot, t_n) \to p(\cdot, t_0) \qquad \text{in} \quad C^1(S^1). \tag{4.10}$$

From (4.7), (4.8), (4.10), and Lemma 2.3, there exists a positive integer N such that

$$u(\cdot, t_n + T) - \sigma_a u(\cdot, t_n) \in \Sigma, \qquad a \in S^1, \tag{4.11a}$$

$$\nu(u(\cdot, t_n + T) - \sigma_a u(\cdot, t_n)) = \nu_0, \qquad a \in S^1 \tag{4.11b}$$

for $n \geqslant N$. Since the parameter a varies in the compact set S^1, one easily sees that the integer N can be chosen independent of $a \in S^1$. In view of (4.11b) and Lemma 2.4(i), we see that

$$\nu(u(\cdot, t+T) - \sigma_a u(\cdot, t)) = \nu_0$$

for $t \geqslant t_N$ and $a \in S^1$. Consequently, by Lemma 2.4(iii), we have

$$u(\cdot, t+T) - \sigma_a u(\cdot, t) \in \Sigma, \qquad t \geqslant t_N, a \in S^1. \tag{4.12}$$

It follows that

$$m(u(\cdot, t+T)) \neq m(u(\cdot, t)), \qquad t \geqslant t_N,$$

where $m(w)$ stands for $\min_{x \in S^1} w(x)$. Without loss of generality we may assume that

$$m(u(\cdot, t+T)) > m(u(\cdot, t)), \qquad t \geqslant t_N.$$

Combining this and (4.9), we see that for any $t \in \mathbb{R}$ there exists $K > 0$ such that

$$m(u(\cdot, t + t_{n+1})) > m(u(\cdot, t + t_n + T)) > m(u(\cdot, t + t_n))$$

for $n \geqslant K$. Letting $n \to \infty$, we obtain

$$m(p(\cdot, t + t_0 + T)) = m(p(\cdot, t + t_0)), \qquad t \in \mathbb{R},$$

which contradicts the previous assertion (4.7). The proof of Lemma 4.3 is complete.

LEMMA 4.4. *Let f and u be as in Lemma 4.2. Suppose $\lim_{n \to \infty} u(\cdot, nT)$ exists in the topology of $C^1(S^1)$. Then there exists a solution $p(x, t)$ of (2.3)–(2.4) with $p(x, t+T) \equiv p(x, t)$ such that*

$$\lim_{t \to \infty} \|u(\cdot, t) - p(\cdot, t)\|_{C^2(S^1)} = 0. \tag{4.13}$$

Proof. Let $\varphi = \lim_{n\to\infty} u(\cdot, nT)$ and p be as constructed in Lemma 4.2 with $t_0 = 0$. By Lemma 4.3, p is T-periodic in t. The convergence (4.13) follows from standard a priori estimates and the continuous dependence of solutions on the initial data. The lemma is proved.

LEMMA 4.5. *Let f and u be as in Lemma 4.2, and let $\tau_0 \in \mathbb{R}/T\mathbb{Z}$. Let φ and $\tilde{\varphi}$ be elements of $\omega(u_0; \tau_0)$. Then either $\varphi = \tilde{\varphi}$ or $\varphi - \sigma_a \tilde{\varphi} \in \Sigma$ for any $a \in S^1$.*

Proof. Assume that $\varphi - \sigma_a \tilde{\varphi} \notin \Sigma$ for some $a \in S^1$. Let p and $\tilde{p}$ be the solutions of (4.5a) with initial values $p(\cdot, t_0) = \varphi$ and $\tilde{p}(\cdot, t_0) = \tilde{\varphi}$, where $t_0 \in \mathbb{R}$ is taken as $\lambda(t_0) = \tau_0$. Define $w = p - \sigma_a \tilde{p}$. Then it is a solution of a linear equation of the form (2.5) with $q(x, t) \equiv 0$ and $r(x, t)$ bounded. We claim that $\varphi = \sigma_a \tilde{\varphi}$. Indeed, if this is not the case, then w is not identically equal to zero and $w(\cdot, t_0) = \varphi - \sigma_a \tilde{\varphi} \notin \Sigma$. Consequently, we have by Lemma 2.4(iii) that

$$v(p(\cdot, t_0 + T) - \sigma_a \tilde{p}(\cdot, t_0 + T)) < v(p(\cdot, t_0 - T) - \sigma_a \tilde{p}(\cdot, t_0 - T)),$$

which, however, is impossible since the functions p and $\tilde{p}$ are T-periodic in t by Lemma 4.3. This contradiction shows $\varphi = \sigma_a \tilde{\varphi}$. By virtue of Theorem B, $\varphi = \sigma_a \tilde{\varphi}$ implies $\varphi = \tilde{\varphi}$. The proof of Lemma 4.5 is complete.

LEMMA 4.6. *Let u and f be as in Lemma 4.2. For each $\tau \in \mathbb{R}/T\mathbb{Z}$, $\omega(u_0; \tau)$ is a compact connected set in $C^1(S^1)$.*

Proof. First note that we can associate with problem (2.3)–(2.4) a semiflow $\Phi = \{\Phi_t\}_{t \geq 0}$ on the product space $C^1(S^1) \times (\mathbb{R}/T\mathbb{Z})$ in a standard way. More precisely, for each $t \geq 0$ the map Φ_t: $C^1(S^1) \times (\mathbb{R}/T\mathbb{Z}) \to C^1(S^1) \times (\mathbb{R}/T\mathbb{Z})$ is given by

$$\Phi_t(\psi, \lambda(s)) = (v(\cdot, t+s; \psi, s), \lambda(t+s)), \qquad \psi \in C^1(S^1), s \in \mathbb{R}, \tag{4.14}$$

where $v(x, t; \psi, s)$ is the solution to the problem

$$v_t = v_{xx} + f(v, t), \qquad x \in S^1, t > s,$$
$$v(x, s) = \psi(x), \qquad x \in S^1.$$

By the periodicity of f in t we have $v(x, t+T; \psi, s+T) \equiv v(x, t; \psi, s)$, therefore the operator Φ_t is well defined. Given a bounded solution $u(x, t)$ to problem (2.3)–(2.4), let

$$\Omega(u_0) = \bigcap_{t \geq 0} \text{closure}\{(u(\cdot, s), \lambda(s)) \mid s \geq t\},$$

where the closure is with respect to the topology of the extended phase space $C^1(S^1)\times(\mathbb{R}/T\mathbb{Z})$. $\Omega(u_0)$ is in fact the ω-limit set of the point $(u_0, \lambda(0))$ under the semiflow Φ. One can easily verify the following relations:

$$\Phi_t(\omega(u_0; \lambda(s))\times\{\lambda(s)\}) = \omega(u_0; \lambda(t+s))\times\{\lambda(t+s)\}, \qquad t, s\in\mathbb{R},$$

$$\Omega(u_0) = \bigcup_{\tau\in\mathbb{R}/T\mathbb{Z}} (\omega(u_0; \tau)\times\{\tau\}). \tag{4.15}$$

By a standard argument, we see that $\Omega(u_0)$ is a compact connected subset of $C^1(S^1)\times(\mathbb{R}/T\mathbb{Z})$, from which the compactness of $\omega(u_0; \tau)$ follows immediately. It remains to show that $\omega(u_0; \tau)$ is connected. Suppose the contrary. Then, for some $\tau_0\in\mathbb{R}/T\mathbb{Z}$ there exist two closed subsets K_1 and K_2 of $C^1(S^1)$ such that

$$K_1\cap K_2 = \varnothing, \tag{4.16a}$$

$$\omega(u_0; \tau_0) = K_1\cup K_2. \tag{4.16b}$$

For each $\tau\in\mathbb{R}/T\mathbb{Z}$, define two subsets of $C^1(S^1)\times\{\tau\}$ by

$$\tilde{K}_i(\tau) = \Phi_t(K_i\times\{\tau_0\}), \qquad i=1, 2,$$

where $t\in\mathbb{R}$ is such that $\lambda(t) = \tau-\tau_0$. It is clear from Lemma 4.3 that the above definition of $\tilde{K}_i(\tau)$ is independent of the choice of $t\in\lambda^{-1}(\tau-\tau_0)$. Let

$$\tilde{K}_i = \bigcup_{\tau\in\mathbb{R}/T\mathbb{Z}} \tilde{K}_i(\tau), \qquad i=1, 2.$$

As is easily seen we have

$$\tilde{K}_i = \{\Phi_t(z) \mid z\in\tilde{K}_i(\tau_0),\ t\in[0, T]\}.$$

By the continuity of the semiflow Φ and the compactness of $\tilde{K}_i(\tau_0)$ and $[0, T]$, the sets $\tilde{K}_i$ $(i=1, 2)$ are compact (hence closed) subsets of $C^1(S^1)\times(\mathbb{R}/T\mathbb{Z})$. Moreover, by (4.16a) and the uniqueness theorem for parabolic equations, we have $\tilde{K}_1\cap\tilde{K}_2=\varnothing$, and (4.15) and (4.16b) imply $\Omega(u_0) = \tilde{K}_1\cup\tilde{K}_2$. But this contradicts the connectedness of $\Omega(u_0)$. The proof of Lemma 4.6 is complete.

Proof of Theorem C. As pointed out in the proof of Lemma A, it suffices to consider problem (2.3)–(2.4). So we assume that f and u are as in Lemma 4.2. By virtue of Lemma 4.4 and Theorem B, what we have to show is that $\omega(u_0; \lambda(0))$ contains precisely one element. Assuming the contrary, we shall derive a contradiction. By Lemma 4.6, $\omega(u_0; \lambda(0))$ is connected.

Therefore, if it contains more than one element, then it must contain infinitely many elements. Choose three distinct elements φ_i $(1 \leqslant i \leqslant 3)$ of $\omega(u_0; \lambda(0))$ and let p_i $(1 \leqslant i \leqslant 3)$ be the solutions of (4.5a) with initial values $p_i(\cdot, 0) = \varphi_i$. By the uniqueness and backward uniqueness theorem for parabolic equations and Lemmas 4.5 and 2.3, we have

$$p_i(\cdot, t) - \sigma_a p_j(\cdot, t) \in \Sigma, \tag{4.17a}$$

$$v(p_i(\cdot, t) - \sigma_a p_j(\cdot, t)) = v_{ij}, \tag{4.17b}$$

for any $t \in \mathbb{R}$, $a \in S^1$, and $i \neq j$, where v_{ij} is an integer independent of t and a. It follows from (4.17a) that $m(p_i(\cdot, t)) \neq m(p_j(\cdot, t))$ if $i \neq j$, therefore we may assume without loss of generality that

$$m(p_1(\cdot, t)) > m(p_2(\cdot, t)) > m(p_3(\cdot, t)), \qquad t \in \mathbb{R}. \tag{4.18}$$

By the definition of $\omega(u_0; \lambda(0))$, we can choose a sequence $t_n \to \infty$ with

$$t_n \equiv 0 \pmod{T} \qquad \text{and} \qquad t_n + T < t_{n+1},$$
$$u(\cdot, t_n) \to p_1(\cdot, 0) \qquad \text{in} \quad C^1(S^1).$$

In view of this and (4.17), and applying Lemma 2.3 and using the periodicity of p_i, we find that for each $a \in S^1$ there exists an integer $N > 0$ such that

$$u(\cdot, t_n) - \sigma_a p_2(\cdot, t_n) \in \Sigma, \tag{4.19a}$$

$$v(u(\cdot, t_n) - \sigma_a p_2(\cdot, t_n)) \equiv v_{12}, \tag{4.19b}$$

for $n \geqslant N$, where v_{12} is as in (4.17b). Since the parameter a varies in the compact set S^1, one easily sees that the integer N can be chosen independent of $a \in S^1$. Consequently, it follows from Lemma 2.4(iii) and (4.19b) that

$$u(\cdot, t) - \sigma_a p_2(\cdot, t) \in \Sigma, \qquad a \in S^1, \, t \geqslant t_N,$$

which implies that $m(u(\cdot, t)) \neq m(p_2(\cdot, t))$ for $t \geqslant t_N$, hence we have either

$$m(u(\cdot, t)) > m(p_2(\cdot, t)), \qquad t \geqslant t_N$$

or

$$m(u(\cdot, t)) < m(p_2(\cdot, t)), \qquad t \geqslant t_N.$$

This, together with (4.18), contradicts the fact that both $\varphi_1 = p_1(\cdot, 0)$ and $\varphi_3 = p_3(\cdot, 0)$ belong to $\omega(u_0; \lambda(0))$. The proof of Theorem C is complete.

5. Autonomous Problems

In this section we consider the case where the equation is autonomous. Therefore the nonlinear term f depends only on u. We assume the following:

(A.4) *$f(u)$ is of class C^1.*

Problem (1.1)–(1.3) now reduces to the form

$$u_t = u_{xx} + f(u), \qquad 0 < x < L,\ t > 0, \tag{5.1}$$

$$u(x, 0) = u_0(x), \qquad 0 < x < L, \tag{5.2}$$

with one of the following boundary conditions:

$$u(0, t) = u(L, t) = 0, \qquad t > 0; \tag{5.3a}$$

$$u_x(0, t) = u_x(L, t) = 0, \qquad t > 0; \tag{5.3b}$$

$$\left.\begin{aligned} u(0, t) &= u(L, t), \qquad t > 0, \\ u_x(0, t) &= u_x(L, t), \qquad t > 0. \end{aligned}\right\} \tag{5.3c}$$

A function $v(x)$ on $[0, L]$ is called an *equilibrium solution* of (5.1)–(5.3) if v satisfies

$$v_{xx} + f(v) = 0, \qquad 0 < x < L, \tag{5.4}$$

together with the boundary conditions corresponding to one of (5.3a)–(5.3c).

The main result in this section is the following:

Theorem D. *Assume $f(u)$ satisfies* (A.4). *Let u be a solution of* (5.1)–(5.3) *such that $\|u(\cdot, t)\|_{C([0, L])}$ remains bounded as $t \to \infty$. Then $u(\cdot, t)$ converges to an equilibrium solution as $t \to \infty$.*

In the cases of the Dirichlet boundary conditions (5.3a) and the Neumann boundary conditions (5.3b), the above theorem is proved in [22, 31]. The case of the periodic boundary conditions (5.3c) is treated in [24].

As mentioned in the Introduction, the point of this section is to present a proof of Theorem D that is different from those in [22, 24, 31]. We shall derive Theorem D from Theorem C. To do this, we need, unfortunately, an additional assumption (A.2)(i) or (A.2)(ii) in the case of the Dirichlet boundary conditions (5.3a).

Proof. Assume (A.2)(i) or (A.2)(ii) if the boundary condition is (5.3a). It is clear that condition (A.3) holds for an arbitrary $T>0$. Therefore, the function $p(x, t)$ in (4.1) must satisfy $p(x, t+T)=p(x, t)$ $(x \in [0, L],\ t \in \mathbb{R})$ for any $T>0$, which implies that $p(x, t)$ is independent of t.

6. Blow-Up Problems

In this section we consider the case where the solution $u(x, t)$ of (1.1)–(1.3) does not exist globally in time, that is, $s(u_0)<\infty$. In this case, the solution blows up in L^∞ norm as $t \uparrow s(u_0)$ (see (1.5)). We make the following assumptions on the nonlinear term f:

(A.5)(i) *$f(u, t)$ is locally Hölder continuous in* $\mathbb{R} \times [0, \infty)$ *and continuously differentiable with respect to u;*

(ii) *$f(0, t) \geqslant 0$ for $t \geqslant 0$.*

(A.6) *There exists a C^2-function F:* $[0, \infty) \to [0, \infty)$ such that

(i) *$F(u)>0$, $F'(u) \geqslant 0$, $F''(u) \geqslant 0$ for $u>0$;*

(ii) *there exist $c>0$ and $M_0>0$ such that*

$$f_u(u, t)\, F(u) - f(u, t)\, F'(u) \geqslant c \cdot F(u)\, F'(u) \tag{6.1}$$

for $u>M_0$, $t \geqslant 0$;

(iii) $\int_1^\infty du/F(u) < \infty$. (6.2)

Definition 6.1. Let $u(x, t)$ be a solution to problem (1.1)–(1.3) such that $s(u_0)<\infty$. We define the *blow-up set* of u by

$$B(u_0) = \left\{ x \in [0, L] \,\middle|\, \begin{array}{l} \text{there exists } x_n \to x \text{ and } t_n \uparrow s(u_0) \\ \text{such that } |u(x_n, t_n)| \to \infty \end{array} \right\}.$$

The points belonging to $B(u_0)$ are called *blow-up points* of u.

The above definition of blow-up set is due to Friedman and McLeod [15]. Conditions similar to (A.6) are also found in [15].

By (1.5) and the compactness of $[0, L]$, $B(u_0)$ is nonempty if $s(u_0)<\infty$. Now we are ready to formulate our main results in this section. We first consider the Neumann and the periodic boundary value problems:

Theorem E. *Let f satisfy* (A.5), (A.6) *and let $u(x, t)$ be a solution to problem* (1.1)–(1.3b) *or* (1.1)–(1.3c) *such that $s(u_0)<\infty$. Assume further that the initial data u_0 is a nonconstant continuous function on* $[0, L]$ *and* *that* $u_0 \geqslant 0$. *Then the blow-up set* $B(u_0)$ *consists of finitely many points and the limit*

$$\varphi(x) = \lim_{t \uparrow s(u_0)} u(x, t) \tag{6.3}$$

exists for every $x \in [0, L] \backslash B(u_0)$. *Moreover,*

(i) φ *is of class* C^2 *in* $[0, L] \backslash B(u_0)$;

(ii) *the cardinal number of* $B(u_0)$ *does not exceed the number of the local maximum points of* u_0.

In the case of the Dirichlet boundary value problem, Theorem E has to be slightly modified:

THEOREM E′. *Let* f *satisfy* (A.5), (A.6) *and let* $u(x, t)$ *be a solution to problem* (1.1)–(1.3a) *such that* $s(u_0) < \infty$. *Assume further that* (A.1)(ii) *or* (A.1)(iii) *in Section* 2 *holds. Then* $B(u_0)$ *is a finite set and the limit* (6.3) *exists for every* $x \in [0, L] \backslash B(u_0)$. *Moreover the assertions* (i), (ii) *of Theorem* E *hold.*

Next we consider the case where $u_0(x)$ is not necessarily nonnegative and the nonlinear term f depends also on u_x. Equation (1.1) is then replaced by

$$u_t = u_{xx} + f(u, u_x, t), \qquad 0 < x < L,\ t > 0. \tag{6.4}$$

We assume the following:

(A.7)(i) $f(u, p, t)$ *is of class* C^1; *moreover* f_{pu}, f_{pp} *and* f_{pt} *exist and are continuous in* $\mathbb{R} \times \mathbb{R} \times [0, \infty)$;

(ii) $f(u, -p, t) = f(u, p, t)$ *and* $f(-u, p, t) = -f(u, p, t)$ *in* $\mathbb{R} \times \mathbb{R} \times [0, \infty)$;

(iii) f_p *is bounded in* $\mathbb{R} \times \mathbb{R} \times [0, \infty)$.

THEOREM F. *Assume that* f *satisfies* (A.7) and (A.6), *where* (6.1) *is understood as*

$$f_u(u, p, t)\, F(u) - f(u, p, t)\, F'(u) \geqslant cF(u)\, F'(u) \tag{6.5}$$

for $u > M_0$, $p \geqslant 0$, $t \geqslant 0$. *Let* $u(x, t)$ *be a solution to problem* (6.4), (1.2), (1.3) *such that* $s(u_0) < \infty$, *with* $u_0(x)$ *being a continuous function on* $[0, L]$. *In the case of the boundary conditions* (1.3b) *or* (1.3c), *assume further that* $u_0(x)$ *is not a constant function. Then* $B(u_0)$ *is a finite set and the limit* (6.3) *exists for every* $x \in [0, L] \backslash B(u_0)$. *Moreover, the statements* (i) *and* (ii) *in Theorem* E *hold, if the term "local maximum points" is replaced by "local extremum points."*

Remark 6.2. Under the assumptions of Theorem F, we have

$$\lim_{t \uparrow s(u_0)} \|u(\cdot, t)\|_{L^\infty(0, L)} = \infty. \tag{6.6}$$

In fact, it is well known that if f satisfies

$$|f(u, p, t)| \leqslant c(u)(1 + p^2), \qquad u \in \mathbb{R}, p \in \mathbb{R}, t > 0 \tag{6.7}$$

for some positive continuous function $c(u)$, then L^∞-boundedness of $u(\cdot, t)$ implies its C^1-boundedness (see Amann [2]). In other words, if the solution blows up in $C^1([0, L])$ norm, then it does so in $L^\infty(0, L)$ norm. And it is clear that (A.7)(iii) implies (6.7).

Remark 6.3. Assumptions (A.5) and (A.6) are satisfied, if, for example,

$$f(u, t) = A(t)\, e^{a(t)u} + b(u, t), \tag{6.8a}$$

$$f(u, t) = A(t)\, u|u|^{q(t)-1} + b(u, t), \tag{6.8b}$$

or

$$f(u, t) = A(t)\, u(\log(1 + |u|))^{r(t)} + b(u, t), \tag{6.8c}$$

where $A(t)$, $a(t)$, $b(u, t)$, $q(t)$, and $r(t)$ are bounded smooth functions in $t \geqslant 0$, $u \geqslant 0$ satisfying

$$A(t) \geqslant A_0, \quad a(t) \geqslant a_0, \quad q(t) \geqslant q_0, \quad r(t) \geqslant r_0,$$

$$|b(u, t)| + |b_u(u, t)| \leqslant b_0,$$

for some $A_0 > 0$, $a_0 > 0$, $q_0 > 1$, $r_0 > 2$, and $b_0 > 0$. In fact, (A.5) is obvious and condition (A.6) can be easily verified by setting $F(u) = e^{\alpha u}$ $(0 < \alpha < a_0)$, $F(u) = u|u|^{\beta - 1}$ $(1 < \beta < q_0)$, and $F(u) = u(\log(1 + |u|))^\gamma$ $(1 < \gamma \leqslant r_0 - 1)$ in the cases (6.8a), (6.8b), and (6.8c), respectively.

Remark 6.4. Lacey [20] shows that if $f(u) = (u + 2)(\log(u + 2))^r$ $(1 < r \leqslant 2)$, then the blow-up set of a solution to problem (1.1), (1.2), (1.3a) contains a nondegenerate interval. This implies that the conclusion of Theorem E′ does not necessarily hold if we drop assumption (A.6)—or, roughly speaking, if $f(u)$ grows too slowly. The same remark also applies to Theorems E and F.

As in the preceding sections, Lemma A of Section 2 (or its modified version, Lemma A′ below) will play an important role in the proof of Theorems E, E′, and F.

LEMMA A′. *Let $f = f(u, p, t)$ satisfy* (A.7) *and let u be a solution of* (6.4), (1.2), (1.3). *Then both of the limits*

$$\lim_{t \uparrow s(u_0)} \operatorname{sgn}(u_x(x, t)) \tag{6.9}$$

and

$$\lim_{t \uparrow s(u_0)} \operatorname{sgn}(u(x, t)) \tag{6.10}$$

exists for every $x \in [0, L]$.

Proof. Arguing as in the proof of Lemma A in Section 2, problem (6.4), (1.2), (1.3), can be converted into a problem on $S^1 = \mathbb{R}/\mathbb{Z}$ of the form

$$u_t = u_{xx} + f(u, u_x, t), \qquad x \in S^1, t > 0, \tag{6.11}$$

$$u(x, 0) = u_0(x), \qquad x \in S^1. \tag{6.12}$$

In fact, in the case of the periodic boundary conditions (1.3c), the equivalence between the two problems—after appropriate rescaling of the variables—is obvious. In the case of the Neumann boundary conditions (1.3b), we extend the solution as in (2.8) and use the symmetry assumption $f(u, -p, t) \equiv f(u, p, t)$ to see that $\tilde{u}$ in (2.8) satisfies an equation of the form (6.4) on $(-L, L)$ together with the periodic boundary conditions at $x = -L, L$. Therefore the problem can be converted into the form (6.11)–(6.12). In the case of the Dirichlet boundary conditions (1.3a), we use the extension (2.9) and the symmetry assumption $f(-u, p, t) \equiv -f(u, p, t)$ to obtain a periodic boundary value problem on $[-L, L]$. Thus it suffices to prove the existence of the limits (6.9) and (6.10) when $u(x, t)$ is a solution of problem (6.11)–(6.12).

First note that Eq. (6.11) is equivariant with respect to the reflection operator $u \mapsto \rho_a u$ and also with respect to the "negative reflection" $u \mapsto -\rho_a u$. The former equivariance follows from the assumption $f(u, -p, t) \equiv f(u, p, t)$ and the latter from $f(-u, p, t) \equiv -f(u, p, t)$ in (A.7)(ii). In view of this, and using the function $w = \rho_a u - u$, which satisfies a linear parabolic equation of the form (2.5), and arguing as in the proof of Lemma A, we can prove the existence of the limit (6.9). Similarly, the existence of the limit (6.10) can be shown by using the function $w = -\rho_a u - u$. The details are omitted. The lemma is proved.

In what follows we first prove Theorem F. Theorems E and E′ will then follow from a similar argument. We need some lemmas:

LEMMA 6.5. *Let f and u be as in Lemma* A′. *In the case of the boundary conditions* (1.3b) *and* (1.3c), *assume further that $u_0(x)$ is a nonconstant function. Then for some $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and $t^* \in [0, s(u_0))$, there exist C^1-curves $\xi_1, \xi_2, \ldots, \xi_m$: $[t^*, s(u_0)) \to [0, L]$ and $\eta_1, \eta_2, \ldots, \eta_n$: $[t^*, s(u_0)) \to [0, L]$ such that*

(i) *for each $t \in [t^*, s(u_0))$,*

$$\xi_1(t) < \xi_2(t) < \cdots < \xi_m(t),$$

$$\eta_1(t) < \eta_2(t) < \cdots < \eta_n(t);$$

(ii) *for each* $t \in [t^*, s(u_0))$,

$$\{x \in [0, L] \mid u(x, t) = 0\} = \{\xi_1(t), \xi_2(t), ..., \xi_m(t)\},$$

$$\{x \in [0, L] \mid u_x(x, t) = 0\} = \{\eta_1(t), \eta_2(t), ..., \eta_n(t)\},$$

moreover, $\{\xi_i(t)\}$ *and* $\{\eta_j(t)\}$ *are simple zeroes of the functions* $u(\cdot, t)$ *and* $u_x(\cdot, t)$, *respectively;*

(iii) *the limits*

$$\alpha_i = \lim_{t \uparrow s(u_0)} \xi_i(t), \qquad \beta_j = \lim_{t \uparrow s(u_0)} \eta_j(t) \tag{6.13}$$

exist for each $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

Proof. We observe that u satisfies a linear parabolic equation of the form (2.5) with $t_0 = 0$, $t_1 = s(u_0)$, $q \equiv 0$, $r(x, t) = f(u, u_x, t)/u(x, t)$, which is locally bounded in $[0, L] \times [0, s(u_0))$ by the condition $f(0, p, t) \equiv 0$, which follows from (A.7)(ii). Moreover, by differentiating Eq. (6.4) with respect to x, we see that $v(x, t) \equiv u_x(x, t)$ also satisfies a parabolic equation of the form (2.5) with $t_0 = 0$, $t_1 = s(u_0)$, $q(x, t) = f_p(u, u_x, t)$, $r(x, t) = f_u(u, u_x, t)$. It follows from Lemma 2.4 that there exists $t^* \in [0, s(u_0))$ such that $u(\cdot, t) \in \Sigma$ and $u_x(\cdot, t) \in \Sigma$ for each $t \in [t^*, s(u_0))$. Applying the implicit function theorem, we obtain C^1-curves $\{\xi_i\}$ and $\{\eta_j\}$ satisfying the assertions (i) and (ii) of this lemma. The convergence (6.13) follows from Lemma A′ immediately. The proof of Lemma 6.5 is complete.

Note that in the above lemma, $m \leqslant \nu(u_0)$ and that n does not exceed the lap number of u_0 (for the definition and properties of lap number, see [23]). In the case where the initial data u_0 is in $C^1([0, L])$, we have $n \leqslant \nu(u_0')$ where $'$ stands for the derivative with respect to x. We also note that the possibility of $\alpha_i = \alpha_{i+1}$ or $\beta_j = \beta_{j+1}$ for some i, j is not excluded in (6.13).

To prove Theorem F, we first show that

$$B(u_0) \subset \{\beta_1, \beta_2, ..., \beta_n\}. \tag{6.14}$$

LEMMA 6.6. *Let* f *and* u *be as in Lemma* 6.5, *and let* $[a, b]$ *be an arbitrary closed interval contained in* $[0, L] \backslash \{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}$. *Then there exists* $t_0 \in [t^*, s(u_0))$ such that $u(x, t)$ *and* $u_x(x, t)$ *do not change sign in the rectangular region* $[a, b] \times [t_0, s(u_0))$.

The proof is immediate from Lemma 6.5.

LEMMA 6.7. *Under the assumptions of Theorem* F, *let* u *and* $Q=[a, b]\times[t_0, s(u_0))$ *be as in Lemma* 6.6. *Assume that*

$$u(x, t)\geqslant 0, \qquad u_x(x, t)\geqslant 0,$$

for $(x, t)\in Q$. *Assume further that* $c\in B(u_0)$ *for some* $c\in(a, b)$. *Then* $\lim_{t\uparrow s(u_0)} u(x, t)=\infty$ *for any* $x\in(c, b]$.

Proof. Choose $d\in(c, b)$ arbitrarily. By the definition of $B(u_0)$, there exist sequences $\{x_k\}$ and $\{t_k\}$ such that $x_k\to c$, $t_k\to s(u_0)$, and $u(x_k, t_k)\to\infty$ as $k\to\infty$. Without loss of generality we may assume that $a<x_k<d$, $t_k\geqslant t_0$ for $k\in\mathbb{N}$. Since $u_x(\cdot, t)\geqslant 0$ on Q, we have

$$u(x, t_k)\geqslant u(x_k, t_k), \qquad d\leqslant x\leqslant b, k\geqslant 1. \tag{6.15}$$

By (A.7)(iii), (6.5), and the assumption that $f(0, 0, t)\equiv 0$ ((A.7)(ii)) along with the fact that $s(u_0)<\infty$, one can easily see that there exist constants $M_1>0$ and $M_2>0$ such that

$$|f(u, p, t)-f(u, 0, t)|\leqslant M_1|p|, \qquad (u, p, t)\in\mathbb{R}\times\mathbb{R}\times[0, \infty);$$
$$f(u, 0, t)\geqslant -M_2 u, \qquad u\geqslant 0, 0\leqslant t<s(u_0).$$

In view of this, we have

$$u_t\geqslant u_{xx}-M_1u_x-M_2u, \qquad (x, t)\in Q. \tag{6.16a}$$

And by the assumption of the lemma, u satisfies

$$u(d, t)\geqslant 0, \qquad u(b, t)\geqslant 0, \qquad t_0\leqslant t<s(u_0). \tag{6.16b}$$

Define

$$v(x, t)=e^{-At}\left(\sin\frac{\pi(x-d)}{b-d}\right)^2, \qquad d\leqslant x\leqslant b, t>0.$$

A simple calculation shows that, for sufficiently large $A>0$,

$$v_t\leqslant v_{xx}-M_1v_x-M_2v, \qquad d\leqslant x\leqslant b, t>0. \tag{6.17a}$$

Furthermore, v satisfies the boundary condition

$$v(d, t)=v(b, t)=0, \qquad t>0. \tag{6.17b}$$

Combining (6.15), (6.16), and (6.17), and using the maximum principle, we see that

$$u(x, t)\geqslant u(x_k, t_k)\cdot v(x, t-t_k), \qquad d\leqslant x\leqslant b, t_k\leqslant t<s(u_0).$$

In view of this and seeing that $s(u_0) < \infty$ and that $v(x, t) > 0$ for $d < x < b$, $t > 0$, we conclude that $\lim_{t \uparrow s(u_0)} u(x, t) = \infty$ for $d < x < b$. It then follows from the assumption $u_x \geqslant 0$ that $\lim_{t \uparrow s(u_0)} u(b, t) = \infty$. Since $d \in (c, b)$ is chosen arbitrarily, the lemma is proved.

LEMMA 6.8. *Let f and u be as in Theorem* F *and let $[a, b]$ be a closed interval contained in $[0, L] \setminus \{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}$. Then $B(u_0) \cap (a, b) = \emptyset$.*

Proof. Assuming the contrary, we shall derive a contradiction. Let $B(u_0) \cap (a, b) \neq \emptyset$.

By Lemma 6.6, there exists $t_0 \in [0, s(u_0))$ such that $u \cdot u_x \neq 0$ for $(x, t) \in [a, b] \times [t_0, s(u_0))$, so without loss of generality we may assume that $u > 0$ and $u_x > 0$ in the region $[a, b] \times [t_0, s(u_0))$. By Lemma 6.7, we can choose some $c \in (a, b) \cap B(u_0)$ such that

$$\lim_{t \uparrow s(u_0)} u(x, t) = \infty \tag{6.18}$$

for any $x \in [c, b]$. If t_0 is taken sufficiently close to $s(u_0)$, we have $u(c, t) > M_0$ for $t_0 \leqslant t < s(u_0)$, hence

$$u(x, t) > M_0, \qquad (x, t) \in Q \equiv [c, b] \times [t_0, s(u_0)),$$

where M_0 is as in (6.5).

Define

$$\zeta(x) = \left(\sin \frac{\pi(x - c)}{b - c} \right)^2, \qquad c \leqslant x \leqslant b, \tag{6.19}$$

and

$$J(x, t) = u_x(x, t) - \varepsilon \zeta(x) F(u(x, t)), \qquad (x, t) \in Q, \tag{6.20}$$

where F is as in (A.6) and ε is a positive constant to be chosen later. A simple computation gives

$$J_t - J_{xx} - q(x, t) J_x - r(x, t) J = \varepsilon \zeta K(x, t) + \varepsilon \zeta F''(u) u_x^2, \tag{6.21}$$

where

$$q(x, t) = f_p(u(x, t), u_x(x, t), t),$$
$$r(x, t) = \varepsilon \zeta(x) F'(u) f_p(u, u_x, t) + f_u(u, u_x, t) + 2\varepsilon \zeta'(x) F'(u),$$

and

$$K(x, t) = f_u F - F' f + \zeta^{-1}(\zeta'' + \zeta' f_p) F + \varepsilon(\zeta f_p + 2\zeta') F F'. \tag{6.22}$$

Considering that $F'(u) \to \infty$ as $u \to \infty$ by virtue of the assumption (A.6), and that $(\zeta'' + \zeta' f_p)/\zeta$ is bounded from below in $(c, b) \times [t_0, s(u_0))$ by virtue of (A.7)(iii), then choosing $\varepsilon > 0$ sufficiently small and using (A.6)(ii) and the fact that $F'' \geqslant 0$, we find that $K(x, t) \geqslant 0$ in Q, hence

$$J_t \geqslant J_{xx} + q(x, t)\, J_x + r(x, t)\, J, \qquad (x, t) \in Q. \tag{6.23}$$

Using $u_x > 0$ and assuming that ε is chosen sufficiently small, we have

$$J(c, t) > 0, \qquad J(b, t) > 0, \qquad t_0 \leqslant t < s(u_0), \tag{6.24a}$$

$$J(x, t_0) > 0, \qquad c \leqslant x \leqslant b. \tag{6.24b}$$

Applying the maximum principle to (6.23) and (6.24), we obtain $J(x, t) > 0$ for $(x, t) \in Q$, or

$$u_x(x, t)/F(u(x, t)) > \varepsilon\zeta(x), \qquad (x, t) \in Q.$$

Integrating this inequality over $c \leqslant x \leqslant b$ yields

$$\int_{u(c,t)}^{u(b,t)} du/F(u) > \varepsilon \int_c^b \zeta(x)\, dx, \qquad t_0 \leqslant t < s(u_0). \tag{6.25}$$

The right-hand side of (6.25) is a positive constant, while the left-hand side tends to zero as $t \uparrow s(u_0)$ by virtue of condition (A.6)(iii) and (6.18). This contradiction shows that $B(u_0) \cap (a, b) = \varnothing$. The proof of Lemma 6.8 is complete.

Proof of Theorem F. We first prove (6.14). Fix $a \in [0, L] \setminus \{\beta_1, ..., \beta_n\}$ arbitrarily. We claim that $a \notin B(u_0)$. We may assume without loss of generality that $a \neq 0$, L, since if $a = 0$ or $a = L$, then by symmetrically extending $u(x, t)$ onto the interval $[-L, 2L]$, we can regard the points 0 and L as interior points of the spatial region on which the solution is defined.

We distinguish two cases: First we consider the case where

$$a \in (0, L) \setminus \{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}.$$

Choose a closed interval

$$[\bar{a}, \bar{b}] \subset (0, L) \setminus \{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}$$

with $a \in (\bar{a}, \bar{b})$. By Lemma 6.8 we have $B(u_0) \cap (\bar{a}, \bar{b}) = \varnothing$, from which it follows that $a \notin B(u_0)$.

Next we consider the case where

$$a = \alpha_i \in (0, L) \setminus \{\beta_1, ..., \beta_n\}$$

for some $1 \leqslant i \leqslant m$. In this case, we can choose a closed interval $[\bar{a}, \bar{b}] \subset (0, L) \backslash \{\beta_1, ..., \beta_n\}$ with $a \in (\bar{a}, \bar{b})$ and

$$[\bar{a}, \bar{b}] \backslash \{a\} \subset (0, L) \backslash \{\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n\}.$$

As we have seen in the first case, it holds that

$$B(u_0) \cap ([\bar{a}, \bar{b}] \backslash \{a\}) = \varnothing. \tag{6.26}$$

By Lemma 6.6, $u_x(x, t)$ does not change sign on $[\bar{a}, \bar{b}] \times [t_0, s(u_0))$ if t_0 is chosen sufficiently close to $s(u_0)$. It follows from this and (6.26) that $a \notin B(u_0)$. Thus, in either case, we have established (6.14).

It remains to show that the limit (6.3) exists and the limit function φ belongs to $C^2([0, L] \backslash B(u_0))$. Choose $x_1 \in [0, L] \backslash B(u_0)$ arbitrarily. As in the above argument, by symmetrically extending the solution $u(x, t)$ on $[-L, 2L]$, we may regard the points 0, L as interior points of the domain of definition of u, so we may assume without loss of generality that

$$x_1 \in (0, L) \backslash B(u_0).$$

Choose a, b, c, d such that $a < c < x_1 < d < b$ and that $[a, b] \subset (0, L) \backslash B(u_0)$. By the definition of $B(u_0)$, $u(x, t)$ is bounded in $D = [a, b] \times [0, s(u_0))$. Choose $t_1 \in (0, s(u_0))$. By the standard L^p and the Schauder's estimates, we see that u, u_x, u_{xx}, u_t are uniformly Hölder continuous in $[c, d] \times [t_1, s(u_0))$. It follows that the limit (6.3) exists for any $x \in [c, d]$ and that the limit function $\varphi(x)$ is of class C^2 on $[c, d]$. Since x_1 is an arbitrary point of $[0, L] \backslash B(u_0)$, the claim is proved. This completes the proof of Theorem F.

Proof of Theorems E *and* E′. The proof of these theorems is quite similar to that of Theorem F. In fact, in view of the assumption $u_0 \geqslant 0$ and (A.5)(ii), and using the fact that $u \not\equiv 0$, and applying the strong maximum principle, we see that $u > 0$ for $0 < x < L$ and t sufficiently close to $s(u_0)$. Combining this and Lemma A, we see that the conclusions (6.9) and (6.10) of Lemma A′ hold. The rest of the proof is now almost the same as (and slightly simpler than) that of Theorem F, so we omit it.

Remark 6.9. In view of the existence of the limit (6.3), one is naturally lead to the question as to what happens to the solution after the blow-up time $s(u_0)$. Of course the solution is defined in the classical sense only for $t \in [0, s(u_0))$. But is there any way to extend the solution in some weaker sense so that it can exist past the blow-up time? Two papers are available relevant to this question. Baras and Cohen [5] study the problem $u_t = u_{xx} + f(u)$ and its approximation problem $u_t = u_{xx} + f_M(u)$, where $f_M(u) = \min\{M, f(u)\}$. Imposing some conditions on f and assuming that a positive solution u of the original problem blows up at $t=s(u_0)$, they show that the approximating solution $u_M(x, t)$ tends to ∞ as $M \to \infty$ for every $x \in (0, L)$ and every $t > s(u_0)$. In the present context, this result can roughly be interpreted that, although the finite limit (6.3) exists at $t=s(u_0)$, the solution $u(x, t)$ becomes ∞ for every $x \in (0, L)$ immediately afterwards. It is therefore unlikely that the solution $u(\cdot, t)$ can be continued past $t=s(u_0)$ in any standard function space. On the other hand, Masuda [21] suggests the possibility of extending the solution past the blow-up time through analytic continuation once t is regarded as a complex variable. He studies the problem $u_t = u_{xx} + u^2$ and, under certain conditions on the initial data u_0, proves that the solution $u(\cdot, t)$ can be continued analytically onto a connected complex region D such that $D \cap \mathbb{R} \supset (0, s(u_0)) \cup (s(u_0)+\delta, \infty)$ for some $\delta > 0$. (As a matter of fact, the result in [21] is stated in a slightly different manner, but it is not difficult to see that the above statement can be derived directly from a theorem in [21] and a standard result on the analyticity of solutions of semilinear parabolic equations.)

References

1. N. D. Alikakos and P. Hess, On stabilization of discrete monotone dynamical systems, *Israel J. Math.* **59** (1987), 185–194.
2. H. Amann, Existence and multiplicity theorems for semilinear elliptic boundary value problems, *Math. Z.* **150** (1976), 281–295.
3. S. Angenent, The zeroset of a solution of a parabolic equation, *J. Reine Angew. Math.* **390** (1988), 79–96.
4. S. Angenent and B. Fiedler, The dynamics of rotating waves in scalar reaction diffusion equations, *Trans. Amer. Math. Soc.* **307** (1988), 545–568.
5. P. Baras and L. Cohen, Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation, *J. Funct. Anal.* **71** (1987), 142–174.
6. P. Brunovský and B. Fiedler, Numbers of zeros on invariant manifolds in reaction-diffusion equations, *Nonlinear Anal.* **10** (1986), 179–193.
7. L. A. Caffarelli and A. Friedman, Blow-up of solutions of nonlinear heat equations, preprint.
8. N. Chafee, Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, *J. Differential Equations* **18** (1975), 111–134.
9. X.-Y. Chen, Uniqueness of the ω-limit point of solutions of a semilinear heat equation on the circle, *Proc. Japan Acad. Ser. A* **62** (1986), 355–337.
10. X.-Y. Chen, H. Matano, and L. Véron, Singularités anisotropes d'équations elliptique semi-linéaires dans le plan, *C. R. Acad. Sci. Paris Sér. I* **303** (1986), 963–966.
11. X.-Y. Chen, H. Matano, and L. Véron, Anisotropic singularities of solutions of nonlinear elliptic equations in $\mathbb{R}^2$, *J. Funct. Anal.*, in press.
12. Y.-G. Chen, Blow-up solutions of a semilinear parabolic equation with the Neumann and Robin boundary conditions, *J. Fac. Sci. Univ. Tokyo Sec. IA*, in press.
13. A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice–Hall, Englewood Cliffs, NJ, 1964.

14. A. FRIEDMAN, On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations, *J. Math. Mech.* **7** (1958), 43–59.
15. A. FRIEDMAN AND B. MCLEOD, Blow-up of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.* **34** (1985), 425–447.
16. H. FUJITA AND Y.-G. CHEN, On the blow-up points and asymptotic behaviours of blow-up solutions to a semilinear parabolic equation, preprint.
17. J. K. HALE, L. T. MAGALHÃES, AND W. M. OLIVA, "An Introduction to Infinite-Dimensional Dynamical Systems—Geometric Theory," Springer-Verlag, New York, 1984.
18. J. K. HALE, Asymptotic behavior and dynamics in infinite dimensions, *in* "Nonlinear Differential Equations" (J. K. Hale *et al.*, Eds.), Research Notes in Math., Vol. 132, Pitman, Boston, 1985.
19. D. KINDERLEHRER AND L. NIRENBERG, Analyticity at the boundary of solutions of nonlinear second-order parabolic equations, *Comm. Pure Appl. Math.* **31** (1978), 283–338.
20. A. A. LACEY, Global blow-up of a nonlinear heat equation, *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986), 161–167.
21. K. MASUDA, Analytic solutions of some nonlinear diffusion equations, *Math. Z.* **187** (1984), 61–73.
22. H. MATANO, Convergence of solutions of one-dimensional semilinear parabolic equations, *J. Math. Kyoto Univ.* **18** (1978), 221–227.
23. H. MATANO, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, *J. Fac. Sci. Univ. Tokyo Sec. IA* **29** (1982), 401–441.
24. H. MATANO, Asymptotic behavior of solutions of semilinear heat equations on S^1, *in* "Nonlinear Diffusion Equations and Their Equilibrium States II" (W.-M. Ni *et al.*, Eds.), MSRI Publications, Vol. 13, (1988), 139–162.
25. X. MORA, Semilinear parabolic problems define semiflows on C^k spaces, *Trans. Amer. Math. Soc.* **278** (1983), 21–55.
26. C. E. MUELLER AND F. B. WEISSLER, Single point blowup for a one-dimensional semilinear heat equation, *Indiana Univ. Math. J.* **34** (1985), 881–913.
27. K. NICKEL, Gestaltsaussagen über Lösungen parabolischer Differentialgleichungen, *J. Reine Angew. Math.* **211** (1962), 78–94.
28. M. H. PROTTER AND H. F. WEINBERGER, "Maximum Principles in Differential Equations," Prentice-Hall, Englewood Cliffs, NJ, 1967.
29. F. B. WEISSLER, Local existence and nonexistence for semilinear parabolic equations in L^p, *Indiana Univ. Math. J.* **29** (1980), 79–102.
30. F. B. WEISSLER, Single point blow-up for a semilinear initial value problem, *J. Differential Equations* **55** (1984), 204–224.
31. T. I. ZELENYAK, Stabilization of solutions of boundary value problems for a second-order parabolic equation with one space variable, *Differential Equations* (translation of *Differentsial'nye Uravneniya*) **4** (1968), 17–22.

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